ON THE RELAXATION OF MATERIAL SYSTEMS

(OB OSVOBOZHDENII MATERIAL'NYKH SISTEM)

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In 1829 Gauss found and described [1] a remarkable property of material systems, namely to move with least constraint. Gauss discovered this property in connection with a transformation of material systems. The Gauss transformation consisted of a relaxation [removal] (the terminology stems from here) of all the constraints of the systems. After Gauss attempts were made to find other forms of relaxation where a property similar to that formulated by Gauss would hold.

In his "Mechanics" Mach noted that the property of least constraint, which holds for systems with complete relaxation, possibly also holds for a relaxation of only a part of their constraints (partial relaxation of material systems). This assumption by Mach was verified by E.A. Bolotov, who proved it for a system with linear differential constraints of the first order [2]. Later on this case was also extended to a system with nonlinear constraints [3].

In 1933 N.G. Chetaev [4] developed a new point of view on the relaxation of material systems. He proposed to apply the term, relaxation, to any system transformation which satisfied some mathematical algorithm (parametric relaxation of material systems). N.G. Chetaev added to his proposition a proof of a corresponding minimum theorem.

In the present paper the problem of the relaxation of a material system is studied from the qualitative point of view. A sufficiently broad qualitative definition of the relaxation of a system is given, and a corresponding minimum theorem is established, after which its mathematical algorithm is derived from the given qualitative definition of the material systems.

A continuous numbering of the geometric and kinematic parameters of the system (coordinates x, velocities x', accelerations x'') and also forces acting on the system is used in this paper. The indices utilized in the paper run through the following values:

$i = 1, \ldots, 3n; \quad \alpha = 1, \ldots, r; \quad \beta = 1, \ldots, s \quad \lambda = 1, \ldots, r'; \quad \mu = 1, \ldots, s.$

Here n is equal to the number of points of the material systems studied, the actual meaning of the values r, s, r', s' will be clear from the text.

1. We shall call a Chetaev system all possible material systems with constraints of the form

$$f_s(t, x_i, x_i') = 0 \tag{1.1}$$

for which "possible displacements" are given by the relations

$$\sum_{i=1}^{3n} \frac{\partial f_s}{\partial x_i'} \delta x_i = 0 \tag{1.2}$$

The Chetaev systems may be outlined completely if one notes that Gauss' principle [4] holds for every one of them, and that these systems are unique among the material systems with constraints of the form (1.1), for which Gauss' principle [5] holds in general.

We shall assume that the system in its given state is more relaxed if in this state the multiplicity of the accelerations which it can acquire in actual motion is enlarged. In relation to this we shall apply the term, relaxation of the material system, to any transformation of it which, without narrowing down the multiplicity of the allowable states of the system, makes the system more relaxed in each one of its given states.

(*Note.* The notion of "enlargement" in this formulation is understood as a "supplement", and thus all states and accelerations, allowable for the basic system, are assumed also to be allowable for the relaxed system.)

Such a qualitative notion of the relaxation of material systems is already completely sufficient for the establishment of the property of Chetaev systems, similar to that formulated by Gauss.

Actually, let A be some Chetaev material system. Then assume that some system B is obtained by a relaxation of the system A, and system B is such that its "allowable displacements" include all "allowable displacements" of system A.

(*Note.* It will be shown below that if system B is also a Chetaev system, then the "allowable displacements" of the original system will also be found among the "allowable displacements" of the latter. The last remark ceases to be essential for this case).

Let us denote by $u_1, \ldots, u_{3n}$ the actual accelerations of system A, by $v_1, \ldots, v_{3n}$ its virtual [4] accelerations. System B is more relaxed, and thus its actual accelerations may differ from the actual accelerations of system A. Let us denote them by $w_1, \ldots, w_{3n}$. Furthermore, we denote by $\delta x_1, \ldots, \delta x_{3n}$ the "possible displacements" of system A and by $\delta^* x_1, \ldots, \delta^* x_{3n}$ the possible displacements" of system B. Finally, we denote by $X_1, \ldots, X_{3n}$ the forces* which act on systems A and B.

We write for each one of the studied systems the fundamental equation of mechanics

$$\sum_{i=1}^{3n} (m_i u_i - X_i)\,\delta x_i = 0, \qquad \sum_{i=1}^{3n} (m_i w_i - X_i)\,\delta^* x_i = 0 \tag{1.3}$$

According to the condition we have

$$\delta x_i \subset \delta^* x_i$$

Thus, equations (1.3) can be written in the following fashion:

$$\sum_{i=1}^{3n} (m_i u_i - X_i)\,\delta x_i = 0, \qquad \sum_{i=1}^{3n} (m_i w_i - X_i)\,\delta x_i = 0$$

By subtracting the second equation from the first, the terms with the forces are eliminated, and we obtain

$$\sum_{i=1}^{3n} m_i (u_i - w_i)\,\delta x_i = 0 \tag{1.4}$$

On the other hand, since system A is a Chetaev system, then

$$\delta x_i = u_i - v_i$$

These equations allow us to rewrite relation (1.4) as

$$\sum_{i=1}^{3n} m_i (u_i - w_i)(u_i - v_i) = 0$$

The last relation reduces to the form

$$A_{uv} - A_{vw} + A_{wu} = 0 \tag{1.5}$$

Here the expression for A_{uv} is of the form

$$\sum_{i=1}^{3n} \frac{m_i}{2} (u_i - v_i)^2$$

and the expressions for A_{vw} and A_{wu} are obtained from A_{uv} by a cyclic permutation of the variables.

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* During relaxation of a material system the forces acting upon it do not change.

The value of A_{uv} is not negative, thus it follows from Equation (1.5) that

$$A_{uw} \leqslant A_{vw}$$

The values A_{uw} and A_{vw} are called, respectively, the measures of deviation of the actual and the virtual motions from the motion of the relaxed system. In the case of complete relaxation this concept coincides with the Gaussian concept of system relaxation. Thus we have the following theorem.

Theorem. If a material system A is a system of the Chetaev type, and a system B is obtained by relaxation of the system A and the "possible displacements" of system B contain all the "possible displacements" of system A, then the law of least deviation of the actual motion of the system from the relaxed system holds (generalized Gauss' principle).

Let us turn now to the derivation of the mathematical algorithm from the qualitative definition of the relaxation of material systems given above. Let us limit ourselves to the case where the relaxation of the Chetaev system to be studied takes place in a class of systems with constraints of the form (1.1).

2. (a) Take some arbitrary material system of the Chetaev type. Denote the Lagrangean coordinates of the system by $q_1, \ldots, q_r$; let $p_1, \ldots, p_s$ be those of the generalized velocities which are assumed to be independent. Then because of the constraint equations of the studied system we can write [3]

$$x_i = a_i(t, q_1, \ldots, q_r), \qquad x_i' = b_i(t, q_1, \ldots, q_r, p_1, \ldots, p_s) \tag{2.1}$$

The right-hand sides of Equations (2.1) are assumed to be differentiable functions of all arguments shown.

(*Note.* Equations

$$x_i' = b_i(t, q_1, \ldots, q_r, p_1, \ldots, p_s) \tag{2.2}$$

can be written in another form.

All coordinates q are actually written out on the right-hand sides of these equations. However, this does not mean yet that they all should be there. In particular cases some and even all q_α may drop out from the right-hand sides of Equations (2.2). With this case in mind and also considering the fact that it is not known beforehand which q_α' will be chosen to be independent, Equations (2.2) can be rewritten as

$$x_i' = b_i(t, q_1, \ldots, q_r, q_1', \ldots, q_r') \tag{2.3}$$

with the essential limitation, however, that the dependent derivatives q_α' drop out of these equations. Such a notation possesses symmetry and this is a very important property.

The dependent values q_α' are known to drop out of Equations (2.3). Further, some q_α may drop out of these Equations. Thus, it is entirely possible that some of the q_α which drop out of equations (2.3) drop out together with their derivatives. In such cases (having changed if necessary the numbering order of the variables q) one can write instead of (2.3) the equations

$$x_i = b_i(t, q_1, \ldots, q_k, q_1', \ldots, q_k') \tag{2.4}$$

where $k < r$, and all coordinates shown in these equations are represented by at least their derivatives. Such a way of representing the multiplicity of the real velocites of the system in the problem of analysing the relaxation of material systems was used by N.G. Chetaev [4].)

Together with system A let us study some arbitrary material system which is obtained by means of a relaxation of system A. We denote it by the letter B. According to the assumption, the constraint equations of system B should be of the form (1.1). Thus, some description of the form (2.1) should hold for system B, as well as for system A. Assume that it is

$$x_i = a_i^*(t, Q_1, \ldots, Q_{r'}), \qquad x_i' = b_i^*(t, Q_1, \ldots, Q_{r'}, P_1, \ldots, P_{s'}) \tag{2.5}$$

The quantities $Q_1, \ldots, Q_{r'}$ in these equations are the Lagrangean coordinates of system B, the parameters $P_1, \ldots, P_{s'}$ are those of the generalized velocities of the system B which are taken to be independent. The functions a_i^*, b_i^* are assumed to be differentiable with respect to all arguments, as in the case of system A.

(b) The multiplicity of the positions of system A depends on r independent parameters and the multiplicity of the positions of system B on r' independent parameters. System B is obtained by a relaxation of A. Thus

$$r' \geqslant r \tag{2.6}$$

On the other hand, since system B is obtained by relaxation of system A the multiplicity of its virtual accelerations in its arbitrary state, which is permissible in the concept of system A, should be at any given instant larger than the multiplicity of the virtual accelerations of system A, which was analyzed in the same state at the same instant. From this follows the relationship between s and s'.

Indeed, after differentiating, with respect to time, the expressions for the velocities x_i' of the basic and correspondingly of the relaxed system, and then expressing in the resulting equations the generalized velocities q' by means of the parameters p and the generalized velocities

Q' by means of the parameters P, we obtain the equations

$$x_i'' = c_i(t, q, p) + \sum_{\beta=1}^{s} c_{i\beta}(t, q, p)\, p_\beta', \quad x_i'' = c_i^*(t, Q, P) + \sum_{\mu=1}^{s'} c_{i\mu}^*(t, Q, P)\, P_\mu' \tag{2.7}$$

for the basic and the relaxed systems, respectively. The derivatives p' and P' do not depend on the values of the remaining parameters. Thus, in every state which is admissible for both systems under consideration, the multiplicities of the virtual accelerations of the basic and the relaxed systems depend on s and s' independent parameters, respectively. Thus, we have

$$r' \geqslant r, \qquad s' > s \tag{2.8}$$

From these inequalities it also follows that within the framework considered, the relaxation of the Chetaev system is always accompanied by a reduction of the number of equations of constraint of this system.

(c) Because of the properties of relaxation, all states that are admissible for system B should be admissible for system A. Thus, for an arbitrary system of values $q_1^*, \ldots, q_r^*, p_1^*, \ldots, p_s^*$ at an arbitrary instant t a system of values $Q_1^*, \ldots, Q_r^*, P_1^*, \ldots, P_s^*$ should exist, such that the following equalities are satisfied:

$$a_i^*(t, Q_1^*, \ldots, Q_{r'}^*) = a_i(t, q_1^*, \ldots, q_r^*)$$
$$b_i^*(t, Q_1^*, \ldots, Q_{r'}^*, P_1^*, \ldots, P_{s'}^*) = b_i(t, q_1^*, \ldots, q_r^*, p_1^*, \ldots, p_s^*)$$

In other words, the functions

$$F_\lambda^*(t, q_1, \ldots, q_r), \qquad \Phi_\mu^*(t, q_1, \ldots, q_r, p_1, \ldots, p_s)$$

should exist where the equations

$$a_i^*(t, F_1^*, \ldots, F_{r'}^*) = a_i(t, q_1, \ldots, q_r)$$
$$b_i^*(t, F_1^*, \ldots, F_{r'}^*, \Phi_1^*, \ldots, \Phi_{s'}^*) = b_i(t, q_1, \ldots, q_r, p_1, \ldots, p_s) \tag{2.9}$$

are satisfied identically for all arguments.

Let us find these functions.

Because of Equations (2.5) the parameters Q are differentiable functions of time and some r' coordinates x (let $x_1, \ldots, x_{r'}$) and the parameters P are differentiable functions of time, coordinates $x_1, \ldots, x_{r'}$ and some s' velocities $x_1', \ldots, x_{r'}'$ (let $x_1', \ldots, x_{s'}'$):

$$Q_\lambda = f_\lambda(t, x_1, \ldots, x_{r'}), \quad P_\mu = \varphi_\mu(t, f_1, \ldots, f_{r'}, x_1', \ldots, x_{s'}') \tag{2.10}$$

We will show that the functions

$$F_\lambda^* = f_\lambda(t, a_1, \ldots, a_{r'}), \qquad \Phi_\mu^* = \varphi_\mu(t, F_1^*, \ldots, F_{r'}^*, b_1, \ldots, b_{s'}) \quad (2.11)$$

satisfy Equations (2.9), depend on all q and p, and are thus the unknowns.

Actually, the following identities hold:

$$x_\lambda \equiv a_\lambda^*(t, f_1, \ldots, f_{r'}), \qquad x_\mu' \equiv b_\mu^*(t, f_1, \ldots, f_{r'}, \varphi_1, \ldots \varphi_{s'})$$
$$(\lambda = 1, \ldots, r', \mu = 1, \ldots, s')$$

They are not violated when x_i and x_i' are replaced by the values from (2.1). Thus

$$a_\lambda^*(t, F_1^*, \ldots, F_{r'}^*) \equiv a_\lambda, \qquad b_\mu^*(t, F_1^*, \ldots, F_{r'}^*, \Phi_1^*, \ldots, \Phi_{s'}^*) \equiv b_\mu$$

On the other hand, the constraint equations of system B will be because of the equations (2.5) and (2.10)

$$x_{r'+1} = a_{r'+1}^*(t, f_1, \ldots, f_{r'}), \ldots$$
$$x'_{s'+1} = b_{s'+1}^*(t, f_1, \ldots, f_{r'}, \varphi_1, \ldots, \varphi_{s'}), \ldots \quad (2.12)$$

System B is obtained by the relaxation of system A. Thus, Equations (2.12) should be satisfied identically if x_i and x_i' are replaced in them by the expressions (2.1), i.e.

$$a_{r'+1}^*(t, F_1^*, \ldots, F_{r'}^*) \equiv a_{r'+1}, \ldots$$
$$b_{s'+1}^*(t, F_1^*, \ldots, F_{r'}^*, \Phi_1^*, \ldots, \Phi_{s'}^*) \equiv b_{s'+1}, \ldots$$

Thus, it is proved that functions (2.11) satisfy the identities (2.9). From these identities it is obvious that the functions F_λ^* should depend on all q and the functions Φ_μ^* on all p. This completes the proof.

The numbers r, s, r', s' are restricted by the inequalities

$$r \leqslant r', \qquad s < s'$$

This means that there are not more parameters q than parameters Q and there are known to be fewer parameters p than parameters P.

Let us supplement the system of parameters q to the number r' and the system of parameters p to the number s' by means of the independent parameters ξ and η, and let us perform the following change of variables in Equations (2.5):

$$Q_\lambda = F_\lambda(t, q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r})$$
$$P_\mu = \Phi_\mu(t, q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r}, p_1, \ldots, p_s, \eta_1, \ldots, \eta_{s'-s}) \quad (2.13)$$

where

$$F_\lambda = F_\lambda^* + \sum_{\gamma=1}^{r'-r} F_{\lambda\gamma}(t, q, p)\,\xi_\gamma$$

$$\Phi_\mu = \Phi_\mu^* + \sum_{\gamma=1}^{r'-r} \Phi_{\mu\gamma}(t, q, p)\,\xi_\gamma + \sum_{\vartheta=1}^{s'-s} \Phi_{\mu\vartheta}^*(t, q, p)\,\eta_\vartheta$$

and where the differentiable functions $F_{\lambda\gamma}$, $\Phi_{\mu\gamma}$, $\Phi_{\mu\theta}$ are so chosen as to satisfy the inequality

$$\frac{\partial(F_1, \ldots, F_{r'}, \Phi_1, \ldots, \Phi_{s'})}{\partial(q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r}, p_1, \ldots, p_s, \eta_1, \ldots, \eta_{s'-s})} \neq 0$$

Because of this inequality, the transformation (2.13) is not degenerate, and thus one can find for every allowable state of system B a corresponding system of the values of parameters q, p, ξ, η. The converse follows directly from Equations (2.13).

By substituting the expressions for Q and P from Equations (2.13) into (2.5) we obtain

$$x_i = A_i(t, q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r}) \tag{2.14}$$
$$x_i' = B_i(t, q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r}, p_1, \ldots, p_s, \eta_1, \ldots, \eta_{s'-s})$$

The right-hand sides of these Equations are such that they pass into a_i and b_i, respectively, if one sets all ξ and η in them equal to zero.

Thus if system B is obtained by a relaxation of system A, then one can write a description of it in the form (2.14).

On the other hand, it is not difficult to verify that if one can write for any arbitrary system B a description of the form (2.14) then this system is relaxed in relation to system A.

Thus we obtained the following mathematical algorithm of the relaxation of a Chetaev system.

In order to perform a relaxation of any material system of the Chetaev type whose multiplicity of the admissible states is given by the equations

$$x_i = a_i(t_1, q_1, \ldots, q_r), \qquad x_i' = b_i(t, q_1, \ldots, q_r, p_1, \ldots, p_s) \tag{2.15}$$

one should transform it in such a fashion that in the new system the multiplicity of the admissible states be given by the equations

$$x_i = A_i(t, q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r})$$
$$x_i' = B_i(t, q_1, \ldots, q_r, \xi_1, \ldots, \xi_{r'-r}, p_1, \ldots, p_s, \eta_1, \ldots, \eta_{s'-s}) \tag{2.16}$$

where ξ and η are additional independent parameters, A_i and B_i are independent quantities in their arguments and transform into the functions a_i and b_i, respectively, if all ξ and η are set equal to zero.

(d) The algorithm of the relaxation of material systems according to Chetaev is included in the transformation of any given material system, for which the description (2.4) is written for the multiplicity of the real velocities of the system, into a form in which this multiplicity would be given by the equations

$$x_i' = b_i(t, q_1, \ldots, q_k, q_1', \ldots, q_k') + \beta_i(t, q_1, \ldots, q_k, \xi_1, \ldots, \xi_e, \xi_1', \ldots, \xi_e')$$

where $\xi_1, \ldots, \xi_e, \xi_1', \ldots, \xi_e'$ are additional independent parameters and their derivatives and the functions β_i are independent functions of the given arguments which vanish when one lets all ξ and ξ' be equal to zero.

Comparing this algorithm with that obtained in this paper, we see that after the parameters p and η in Equations (2.16) are expressed in terms of parameters q, ξ and their derivatives, the difference between these algorithms reduces to the functions of different degrees of generality used in the right-hand sides of Equations (2.16).

3. Let us return now to the problem of the "possible displacements" of the basic and the relaxed systems in the case where both are Chetaev systems.

As already noted above, the following equations hold for the Chetaev system

$$\delta x_i = u_i - v_i \tag{3.1}$$

The values u_i and v_i are components of the actual and of any virtual accelerations of the system, respectively.

According to (2.7) they are expressed by the equations

$$u_i = c_i(t, q, p) + \sum_{\beta=1}^{s} c_{i\beta}(t, q, p)\, p_{\beta 0}', \qquad v = c_i(t, q, p) + \sum_{\beta=1} c_{i\beta}(t, q, p)\, p_\beta'$$

where $p_{\beta 0}'$ denote the magnitudes of the values of p_β' corresponding to the actual motion.

Thus

$$\delta x_i = \sum_{\beta=1}^{s} c_{i\beta}\sigma_\beta \tag{3.2}$$

where $\sigma_\beta = p_{\beta 0}' - p_\beta$ are, obviously, independent. Assuming that

$$c_{i\beta} = \frac{\partial b_i}{\partial p_\beta}$$

we rewrite equations (3.2) in the form

$$\delta x_i = \sum_{\beta=1}^{s} \frac{\partial b_i}{\partial p_\beta} \sigma_\beta \tag{3.3}$$

The following theorem holds.

Theorem. If the basic and the relaxed system are both Chetaev systems then the "possible displacements" of the basic system are always among the "possible displacements" of the relaxed system.

Actually, let systems A and B be the basic and the relaxed systems, respectively. Assume that for the first system the description (2.15) and for the second one (2.16) holds. Then the following equations will hold for the "possible displacements" of systems A and B, respectively:

$$\delta x_i = \sum_{\beta=1}^{s} \frac{\partial b_i}{\partial p_\beta} \sigma_\beta, \qquad \delta^* x_i = \sum_{\beta=1}^{s} \frac{\partial B_i}{\partial p_\beta} \sigma_\beta + \sum_{\gamma=1}^{s'-s} \frac{\partial B_i}{\partial \eta_\gamma} \pi_\gamma$$

In the second group of equations, let all ξ, η and also π be equal to zero. Denote this substitution by square brackets; then

$$[\delta^* x_i] = \sum_{\beta=1}^{s} \left[\frac{\partial B_i}{\partial p_\beta}\right] \sigma_\beta = \sum_{\beta=1}^{s} \frac{\partial [B_i]}{\partial p_\beta} \sigma_\beta = \sum_{\beta=1}^{s} \frac{\partial b_i}{\partial p_\beta} \sigma_\beta = \delta x_i$$

Thus, if the conditions $\xi = \eta = \pi = 0$ are satisfied, then the expressions for the "possible displacements" of the relaxed system transform into the expressions for the "possible displaoements" of the basic system. This completes the proof.

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